VINOD CHANDRU
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Sec 111

Fall 03 Final Exam - Hutchings

4. Calculate $\iint_S F \cdot dS$ where S is the portion of the paraboloid $z = 9 - x^2 - y^2$ with $z \geq 0$, oriented using the upward pointing normal, and $F = \langle x, y, z \rangle$.

Solution: $F = xi + yj + zk$

Using the parametric representation

$r(\phi, \theta) = 3\sin\phi\cos\theta\, i + 3\sin\phi\sin\theta\, j + 3\cos\phi\, k \qquad 0 \leq \phi \leq \frac{\pi}{2} \quad (\because z \geq 0)$

$0 \leq \theta \leq 2\pi$

we have $F(r(\phi,\theta)) = 3\sin\phi\cos\theta\, i + 3\sin\phi\sin\theta\, j + 3\cos\phi\, k$

$$r_\phi \times r_\theta = \begin{vmatrix} i & j & k \\ 3\cos\phi\cos\theta & 3\cos\phi\sin\theta & -3\sin\phi \\ -3\sin\phi\sin\theta & 3\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= 9\sin^2\phi\cos\theta\, i + 9\sin^2\phi\sin\theta\, j + 9\sin\phi\cos\phi\, k$$

$$\therefore F(r(\phi,\theta)) \cdot (r_\phi \times r_\theta) = 27\sin^3\phi\cos^2\theta + 27\sin^3\phi\sin^2\theta + 27\sin\phi\cos^2\phi$$

$$= 27\sin^3\phi + 27\sin\phi\cos^2\phi$$

$$= 27\sin\phi\,(\sin^2\phi + \cos^2\phi)$$

$$= 27\sin\phi$$

$$\therefore \iint_S F \cdot dS = \iint_D F \cdot (r_\phi \times r_\theta)\, dA = \int_0^{2\pi}\int_0^{\pi/2} 27\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} -27\cos\phi \Big|_0^{\pi/2} d\theta$$

$$= \int_0^{2\pi} 27(\cos 0 - \cos \pi/2)\, d\theta$$

$$= 2\pi \times 27 = \boxed{54\pi}$$